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Comparisons of Galerkin and Finite Difference Methods for Solving Highly Nonlinear Thermally Driven Flows

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The equations of motion for a high Prandtl number Boussinesq fluid in a square 2-D cavity with side-wall heating and cooling and perfectly conducting end walls have been solved by means of Galerkin as well as ADI (alternating-direction-implicit) finite difference methods for Rayleigh numbers up to 8×10^6 and two angles of tilt. The finite difference solutions for the conductive flux at the heated wall converge monotonically from above with increasing numbers of mesh points; whereas, the Galerkin solutions converge from below and undergo modest oscillations with long period as additional terms are included. The nearly quiescent core and associated hydrodynamic boundary layers are, for given numbers of mesh points/terms, better represented by the finite difference method. With increasing precision in the wall heat flux and/or shear, the computational costs for both methods become comparable; however, for errors in excess of 2–3 %, the Galerkin method is more economical.

INTRODUCTION

It long has been recognized that numerical integration of the conservation equations for mass, momentum, energy, and species in two- and three-space dimensions becomes increasingly more difficult as the Re/Ra numbers for forced/natural convective flows increase. The principal difficulty derives from the effects of increased nonlinearity and coupling via the advective terms which, for various finite difference schemes, lead to computational instability. Such instability may arise, for example, when a given scheme fails to conserve mean vorticity and either mean kinetic energy or square vorticity [1].

Historically, early approaches approximated the governing flow equations (in terms of either the primitive variables $\mathbf{v}$, P or stream function-vorticity) by means of central difference analogs, the resulting algebraic equations being solved iteratively by successive substitution techniques [2–5]. Such methods worked well at low Reynolds numbers but failed to converge at higher Reynolds numbers. Although the stability problem may to some extent be alleviated by use of under-relaxation [6], the cost proves to be prohibitive.

In the search for economical methods which are stable at all Re/Ra numbers, several investigators, following an observation by Courant *et al.* [7], abandoned central differencing of the advective terms in favor of so-called "upwind-differencing" [8, 9, 10]. Although the method has proved to be unconditionally stable, appreciable loss of accuracy accrues as Re/Ra become large due to false-diffusion effects associated with first order truncation errors in differencing the advective terms [11]. Further discussion and an error analysis of the upwind scheme is given in [12].

In an effort to retain precision at higher Re/Ra numbers, an alternative school of thought has emerged in which conditionally-stable, fully centrally differenced analogs have been devised for which coupled sets of nonlinear algebraic equations are solved by means of alternating-direction-implicit (ADI) techniques. The method has been successfully applied to shear- and buoyancy-driven cavity flows at $Re \leqslant 2000$ and $Ra \leqslant 360{,}000$, respectively [13], to entrance region flows ($Re \leqslant 75{,}000$) [14], to flows in a cylindrical enclosure induced by high rates of surface evaporation–condensation in the presence of noncondensable [15], and to thermally driven flow of nonnewtonian fluid [16]. In [13], some difficulty in obtaining solutions at $Re > 2000$ was cited; however, in recent unpublished work by the first author, solutions were obtained up to $Re \sim O(10^4)$ and it was found that the ADI method is markedly superior to upwind differencing, in terms of both accuracy and efficiency, as the Reynolds number becomes large. Similar conclusions were reached in [13]. More recently, a hybrid algorithm has been proposed [17] involving combination of the upwind and central-difference schemes. Application of the algorithm to energy transfer across a simple one-dimensional Couette-type flow proves to be superior to either scheme.

In contrast to integrating the conservation equations by means of finite difference methods, a markedly different approach is emerging in which dependent variables are expanded in sequences of spatially dependent functionals (with the necessary property that they form a complete set), the coefficients in the resulting series being extracted from simple integrations over the domain of interest. This approach, often referred to as the Galerkin method, offers considerable flexibility since it does not require that the set of "trial" functions be orthogonal and appreciable freedom exists with respect to ordering the sequence (see e.g. [18, 19]). Thus, functions may be selected which satisfy not only the boundary conditions but also, depending on the insight and skill of the analyst, other features of a given problem as well. Furthermore, unlike finite difference methods, evaluation of wall fluxes is exact within the accuracy of the internal field results. However, little is known about the method's convergence properties and storage requirements for all but the largest of digital computers may pose a problem if appreciable numbers of terms are required to effect a desired accuracy.

An early application of the Galerkin-type approach involved the use of orthogonal polynomials as trial functions in a two-dimensional analysis of thermal convection in enclosed plane gas layers [20], with $500 \leqslant \mathrm{Ra} \leqslant 10^4$. Results were obtained for a vertical square cavity using modest numbers of terms in the series expansion, convergence being presumed when additional terms produced negligible changes in the results. Later applications include an analysis of 2-D Bénard convection [21], for which the trial functions were eigenfunctions of the stability problem, and a study of 2-D motions in a horizontal layer of a high Pandtl number fluid heated from below [22], using from 12 to 42 terms in a Fourier expansion for temperature with $\mathrm{Ra}_c < \mathrm{Ra} \leqslant 30{,}000$.

The purpose of the present contribution is to compare Galerkin and finite difference methods of analysis for highly nonlinear, thermally driven flows. (As implied above, it appears that the conditionally stable ADI approach is the most promising finite difference method available, and it has been chosen for the present study.) The physical situation is depicted in Fig. 1, where the model problem is

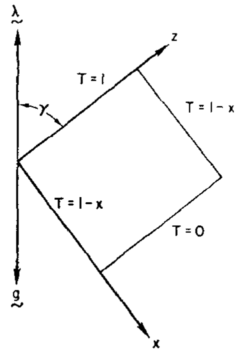

FIG. 1. Schematic of physical situation.

one of steady 2-D motion of a large Prandtl number fluid in a square cavity with perfectly conducting end walls at $z = 0, 1$. Steady 2-D motion at $\mathrm{Ra} \sim 10^7$ for a nearly square cavity has been experimentally observed by Elder [23]. Further, Busse [24] has recently argued that the onset of nonstationary convection occurs at Rayleigh numbers which increase approximately linearly with Prandtl number. The assumption of steady motion at large Rayleigh number for a high Prandtl number fluid appears, therefore, to be justified. Perfectly conducting, as opposed to adiabatic, end walls are considered because the associated thermal boundary layers for the former provide a more severe test of the methods. Due to the centro-symmetric character of the problem about $x = 1/2$, only the half-plane $0 \leqslant x \leqslant 1/2$ need be considered.

ANALYSIS

For steady, two-dimensional, thermally driven flow of a Boussinesq fluid within an enclosure, dimensionless forms of the governing conservation equations, with Pr $\rightarrow\infty$, may be written as

$$\nabla^4\psi - \cos\gamma\frac{\partial T}{\partial x} - \sin\gamma\frac{\partial T}{\partial z} = 0, \tag{1}$$

$$\frac{\partial\psi}{\partial z}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial z} = \mathrm{Ra}^{-1}\nabla^2 T, \tag{2}$$

where $u = \partial\psi/\partial z$, $w = -\partial\psi/\partial x$, and Rayleigh number is defined as $\mathrm{Ra} = \alpha g\,\Delta T\,d^3/\nu\kappa$; and the nondimensionalization has been effected with respect to length scale d, velocity scale Ra κ/d, and temperature scale ΔT. Thermal boundary conditions are given in Fig. 1; for the flow, $\psi = \partial\psi/\partial n = 0$ at the solid walls.

Galerkin Method

A detailed exposition of the method is given in [18, 19, 25] wherein, on expanding T and ψ in terms of trial functions T_m and ψ_m

$$T(x, z) = (1 - x) + \sum_{m=1}^{N} A_m T_m(x, z), \tag{3}$$

$$\psi(x, z) = \sum_{m=1}^{N} B_m\psi_m(x, z), \tag{4}$$

substituting the resulting expansions in Eqs. (1) and (2), and integrating the result over the unit square, there obtains

$$\sum_{m=1}^{N} B_m\langle\psi_n\nabla^4\psi_m\rangle - \sum_{m=1}^{N} A_m\left\langle\psi_n\left(\cos\gamma\frac{\partial T_m}{\partial x} + \sin\gamma\frac{\partial T_m}{\partial z}\right)\right\rangle = -\cos\gamma\langle\psi_n\rangle, \tag{5}$$

$$\sum_{m=1}^{N} B_m\left\langle T_n\left[-\frac{\partial\psi_m}{\partial z} + \sum_{k=1}^{N} A_k\left(\frac{\partial\psi_m}{\partial z}\frac{\partial T_k}{\partial x} - \frac{\partial\psi_m}{\partial x}\frac{\partial T_k}{\partial z}\right)\right]\right\rangle$$

$$= \mathrm{Ra}^{-1}\sum_{m=1}^{N} A_m\langle T_n\nabla^2 T_m\rangle. \tag{6}$$

Here, $\langle\phi\rangle$ denotes the double integration $\int_0^1\int_0^1 \phi\,dx\,dz$, A_m and B_m are undetermined Galerkin coefficients, and the integrations are performed with respect to weighting functions ψ_n and T_n, respectively. With $n = 1, 2,\ldots, N$, Eqs. (5) and (6) reduce to an algebraic problem in the $2N$ unknown coefficients A_m and B_m. Denoting the integrals as matrices (or vectors), as required, and using the summation convention

$$B_m I^{(1)}_{mn} + A_m I^{(2)}_{mn} = I^{(3)}_n, \tag{7}$$

$$B_m(I^{(4)}_{mn} + A_k I^{(5)}_{mkn}) = \mathrm{Ra}^{-1} A_m I^{(6)}_{mn} . \tag{8}$$

From Eq. (7),

$$B_k = (I^{(3)}_n - A_m I^{(2)}_{mn})\, I^{(1)^{-1}}_{nk}. \tag{9}$$

Substituting Eq. (9) into (8), the algebraic problem for the coefficients A_m assumes the form

$$A_i A_j M_{ijk} + A_i D_{ij} = E_j . \tag{10}$$

Trial functions T_m and ψ_m have been devised [25, 26] which satisfy the boundary conditions and, additionally, take advantage of the centrosymmetric character of the problem. For $m \leqslant N/2$

$$\begin{aligned} T_m &= \sin(2J_m\pi x)\sin[(2K_m - 1)\,\pi z], \\ \psi_m &= C_{J_m}(x - 1/2)\, C_{K_m}(z - 1/2), \end{aligned} \tag{11}$$

while, for $m > N/2$,

$$\begin{aligned} T_m &= \sin[(2J_m - 1)\,\pi x]\sin(2K_m\pi z), \\ \psi_m &= S_{J_m}(x - 1/2)\, S_{K_m}(z - 1/2), \end{aligned} \tag{12}$$

where

$$C_n(\phi) = \frac{\cosh(\lambda_n\phi)}{\cosh(\lambda_n/2)} - \frac{\cos(\lambda_n\phi)}{\cos(\lambda_n/2)} \tag{13}$$

and

$$S_n(\phi) = \frac{\sinh(\mu_n\phi)}{\sinh(\mu_n/2)} - \frac{\sin(\mu_n\phi)}{\sin(\mu_n/2)} . \tag{14}$$

The λ_n and μ_n are roots of the characteristic equations $\tanh(\lambda/2) + \tan(\lambda/2) = 0$ and $\coth(\mu/2) - \cot(\mu/2) = 0$, respectively. In selecting combinations of integers $\{J_m, K_m\}$ to form the sequences T_m and ψ_m, two choices are available, either

$$J_m \leqslant M \qquad \text{and} \qquad K_m \leqslant M$$

or

$$J_m + K_m \leqslant M.$$

The former will be denoted "full-mode" selection (FM) whereas the latter will be termed "diagonal-mode" selection (DM). Both choices were considered and, as might be expected due to inclusion of some higher modes, the latter selection gave better convergence properties.

Solution of the algebraic equations (10) for the Galerkin coefficients was obtained by the Newton–Raphson method. This required a reasonably close estimate of the true solution to ensure convergence. At low Ra, the estimate was extracted from the linearized form of Eq. (10), upon setting $M_{ijk} = 0$, and solving by matrix inversion techniques. For larger Ra, the linearized solution is a poor approximation to the full equations and the solution at a previous Ra number served as a first guess in the iterative procedure. Starting at Ra = 10,000–300,00, with the linearized solution as a first estimate, succeeding solutions were obtained by increasing Ra by factors no greater than 2 or 3. Using this procedure together with a relaxation parameter of 0.5 on the first few iterations, a solution at each succeeding Ra number was obtained in 10–15 iterates.

Finite Difference Procedures

Reducing Eq. (1) to a pair of second-order equations in stream function-vorticity, rewriting Eq. (2) in full divergence form, and approximating first and second derivatives by second-order-correct central-difference analogs, one cycle of the ADI scheme for the resulting algebraic problem assumes the form

$$D_n \phi_{n,i,j}^{(k+1/2)} = A_{n,i,j} \phi_{n,i+1,j}^{(k+1/2)} + B_{n,i,j} \phi_{n,i-1,j}^{(k+1/2)} + C_{n,i,j}^{(k)} \qquad [(i = 2, 3,\ldots, I), j = 2, 3,\ldots, J-1], \tag{15}$$

$$D_n \phi_{n,i,j}^{(k+1)} = A_{n,i,j} \phi_{n,i,j+1}^{(k+1)} + B_{n,i,j} \phi_{n,i,j-1}^{(k+1)} + C_{n,i,j}^{(k+1/2)} \qquad [(j = 2, 3,\ldots, J-1), i = 2, 3,\ldots, I], \tag{16}$$

where $I - 1 = 0.5/\Delta x$, $J - 1 = 1/\Delta z$, and $\Delta x = \Delta z = \Delta$; and, for example, the x-direction half cycle is defined as follows.

n	ϕ_n	D_n	$A_{n,i,j} - 1$	$B_{n,i,j} - 1$	$C_{n,i,j}^{(k)} - \Delta^2 \dfrac{\partial^2 \phi_n^{(k)}}{\partial z^2} - \sigma_{\phi_n} \phi_{n,i,j}^{(k)}$
1	ψ	$2 + \sigma_\psi$	0	0	$\Delta^2 \omega_{i,j}$
2	ω	$2 + \sigma_\omega$	0	0	$\Delta^2(\cos\gamma(\partial T/\partial x) + \sin\gamma(\partial T/\partial z))_{i,j}$
3	T	$2 + \sigma_T$	$-(\mathrm{Ra}/4)(\psi_{j+1} - \psi_{j-1})_{i+1}$	$(\mathrm{Ra}/4)(\psi_{j+1} - \psi_{j-1})_{i-1}$	$-\mathrm{Ra}\, \Delta^2(\partial/\partial z)(T(\partial\psi/\partial x))_{i,j}$

In addition to the ADI iteration parameters $\sigma_n = \Delta^2/\Delta\tau_n$, field- and boundary-weighting parameters are introduced such that

$$\phi_{n,i,j}^{(k+1)} = \sigma_{\phi_n,F}\phi_{n,i,j}^{(k)} + (1 - \sigma_{\phi_n,F})\,\phi_{n,i,j}^{(k+1)}, \tag{17}$$

following each ADI cycle and

$$\omega_s^{(l+1)} = \sigma_{\omega,s}\omega_s^{(l)} + (1 - \sigma_{\omega,s})\,\omega_s^{(l+1)}, \tag{18}$$

following each recalculation of the wall vorticity. Following [14], ω_s was extracted from the second-order correct expression

$$\omega_s = -(\psi_{s+1} + 4\psi_{s+2} - \psi_{s+3})/(2\Delta)^2 \tag{19}$$

where $s, s+1, \ldots,$ denote locations at the wall, one point in from the wall, etc.

The overall scheme for converging the coupled algebraic problem is demonstrated in Fig. 2. Starting with the fluid at rest, solutions were obtained sequentially

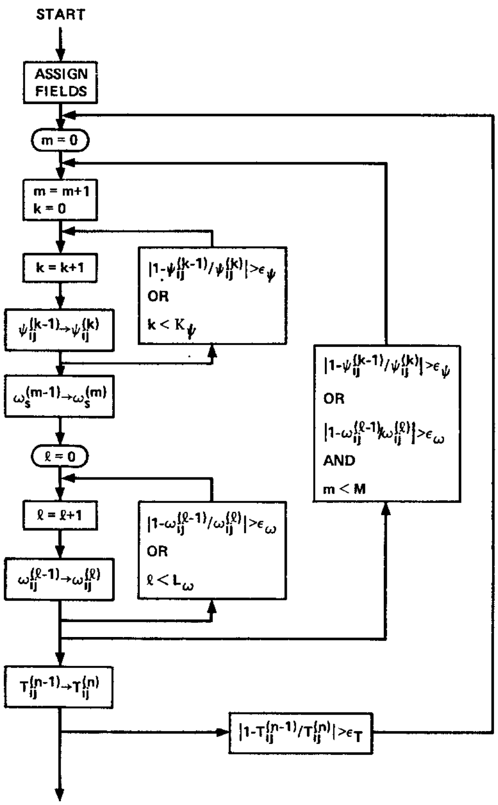

FIG. 2. Flow chart for finite difference solution.

at successive Rayleigh numbers using the converged solution at a previous Rayleigh number as "initial" condition for the next. The number of passes K_ψ, L_ω, and M in the fluid mechanics loops per pass through the energy equation, as well as the σ's, were treated as parameters for optimizing the iterative scheme.

Results and Discussion

Convergence Properties

Comparisons of convergence characteristics for the Galerkin and ADI finite difference methods for integrating the equations governing thermally driven flow of a high Prandtl number Boussinesq fluid in a square 2-D cavity are presented in Figs. 3–9. The number of terms in the Galerkin expansion as well as strips across the cavity are denoted by N; for the Galerkin results, full- and diagonal-mode bases of selection are denoted by FM and DM, respectively. Although results were obtained at Ra $\times\ 10^{-6} = 0.003, 0.01, 0.03, 0.1, 0.3, 0.8, 1.2, 2.0, 4.0$, and 8.0, only results at high Ra will be discussed here.

The effects of N on wall heat transfer are illustrated in Figs. 3 and 4, where

$$Nu = -\int_0^1 \frac{\partial T}{\partial x}\, dz \bigg|_{x=0}.$$

At $\gamma = 0°$, the finite difference results for $N = 100$ are essentially exact, giving $Nu \doteq 0.112\ \mathrm{Ra}^{0.294}$, with the Nusselt number at Ra $= 8 \times 10^6$ being $\sim 2\,\%$ high as estimated by means of Δ^2-extrapolation. At $\gamma = 60°$, the results at Ra $= 1.2 \times 10^6$ are essentially converged using $N = 60$ strips or 72 DM terms. The effects of N on Nu are more clearly displayed in Fig. 4, where it is seen that the finite difference solutions converge monotonically from above whereas the Galerkin solutions converge from below and appear to oscillate as N becomes large. Thus, the common practice whereby Galerkin analyses are presumed to have converged when added terms produce negligible change in a given property of a solution may be misleading. Furthermore, it would seem that the DM basis of selection is superior to FM (compare dashed curves in Fig. 4); however, the DM basis leads to earlier and more pronounced overshoot in Nu, and the merit of DM-selection is obscured. Of further interest is the result that for a given accuracy in Nu, with the constraint that the thermal and flow fields be self-consistent, $\partial N/\partial \mathrm{Ra}$ is appreciably larger for the Galerkin as opposed to the finite difference precedure. For example, it is estimated that convergence of Nu to within $2\,\%$ at 8×10^6 would require on the order of 200 DM terms.

The effects of N on the flow field are illustrated in Figs. 5–7 in terms of the velocity profiles at mid-height ($w = -\partial\psi/\partial x$) and midplane ($u = \partial\psi/\partial z$). The

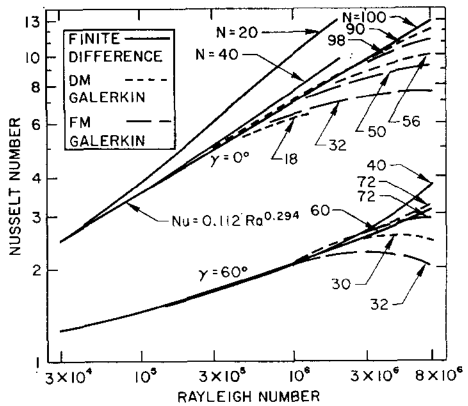

FIG. 3. Convergence characteristics of Nusselt number at heated wall with increasing Rayleigh number.

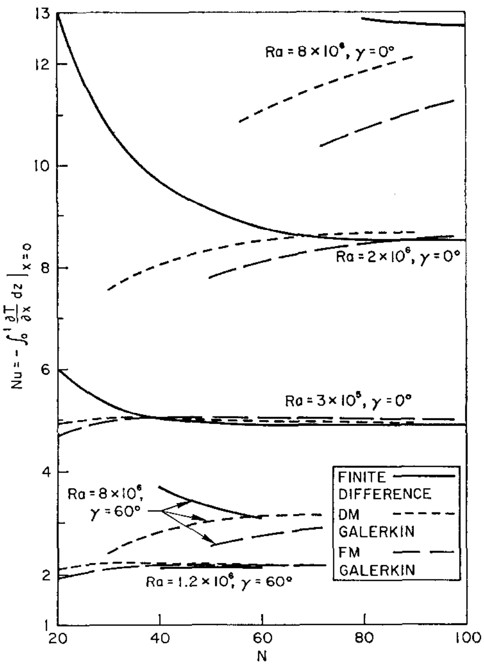

FIG. 4. Convergence characteristics of Nusselt number at heated wall with increasing number (N) of strips/terms.

finite difference results, as suggested by Fig. 4, overpredict the extremum in w and u, but tend to well represent boundary layer locations and the nearly stagnant core of the flow (see Elder's results [23] for adiabatic end walls). The Galerkin results, on the other hand, tend to misrepresent the general character of the boundary layer flows, perhaps as a consequence of the inability of the truncated series to handle well the nearly quiescent core region (see Figs. 5 and 7). Again, the diagonal mode of selection appears to be superior. The results obtained using either method are still insufficiently precise to resolve even qualitatively the fine detail of the core flow (see e.g. Fig. 7).

The effects of N on the thermal field are illustrated in Figs. 8 and 9. At midheight, the thermal boundary layers along the hot wall are less sharply defined than are the hydrodynamic boundary layers (compare Fig. 8 with Figs. 5 and 6). This

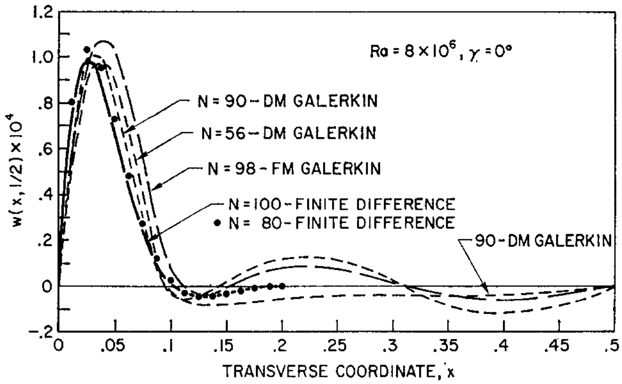

FIG. 5. Velocity profile at midheight, $\gamma = 0°$.

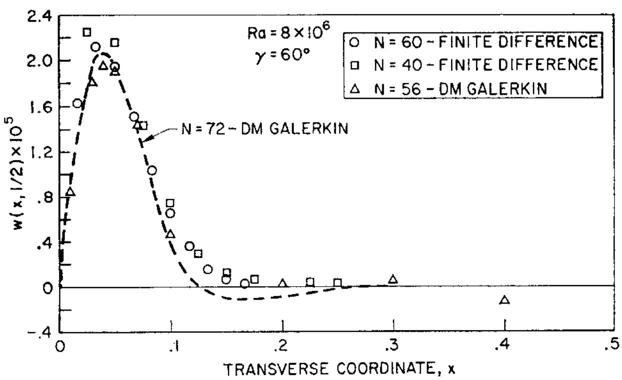

FIG. 6. Velocity profile at midheight, $\gamma = 60°$.

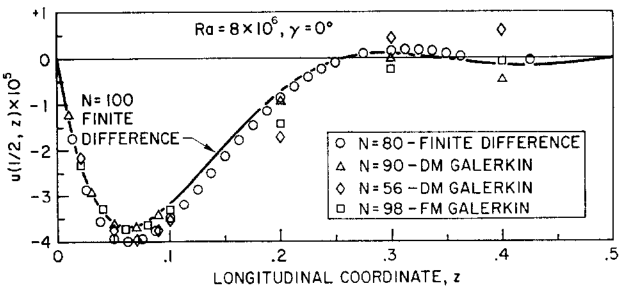

FIG. 7. Velocity profile at midplane, $\gamma = 0°$.

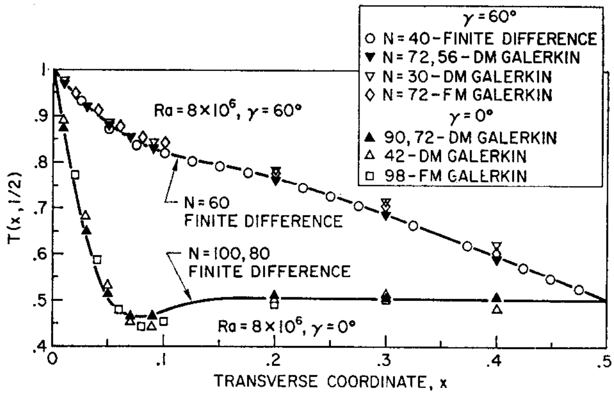

FIG. 8. Temperature profile at midheight, $\gamma = 0°, 60°$.

observation, coupled with the first-order effect of energy advection in the core, accounts for the relatively greater success of the Galerkin method in predicting wall heat transfer as opposed to wall shear.

Computational Efficiency

The preferred basis for assessing the relative efficiencies of the methods would entail direct quantitative comparisons of the computational times required to obtain solution, of a priori prescribed accuracies, for the sequence

$$\text{Ra} = 3 \times 10^3 \rightarrow 8 \times 10^6.$$

Since the computational costs for such an approach would have been prohibitively high, only qualitative inferences, based on the data in Fig. 4 for the $\gamma = 0$ case, will be made.

For a 2% "error" in Nu at Ra $= 8 \times 10^6$, relative to the "true" solution as obtained by Δ^2-extrapolation of the finite difference solutions at $N = 80$ and 100, it is found that the computational times for the Ra sequence are comparable. At reduced accuracies, the Galerkin procedure proves to be increasingly more efficient, requiring about 1/2 to 1/3 as much time for errors of the order of 10%. However, as discussed earlier, the Galerkin solutions tend to misrepresent more seriously the hydrodynamics despite comparable accuracies in wall heat transfer.

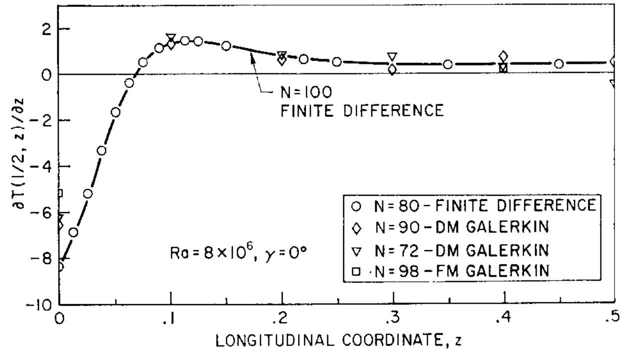

FIG. 9. Temperature gradient profile at midplane, $\gamma = 0°$.

Of note with respect to the ADI finite difference method is the increased difficulty encountered in obtaining steady solutions at high Ra. It was found that increasingly larger ADI parameters (σ_ϕ's) were required to eliminate spurious oscillatory transients with long period as Ra $\to 8 \times 10^6$. Over the range Ra $= 3 \times 10^3 \to 8 \times 10^6$, $\sigma_\psi(= \sigma_\omega)$ and σ_T ranged approximately from 0.1 to 10 and from 1 to 100, respectively. (The field weighting parameters $\sigma_{\phi_n, F}$ as well as the wall vorticity factor $\sigma_{\omega,s}$ were fixed at ~ 0.2 and 0.85, respectively.) Furthermore, it proved useful to promote intermediate "solutions" for the flow field which were in-phase with the thermal field. This was done by increasing the number of passes K_ψ, L_ω, and M through the fluid mechanics loops per pass through the energy equation as Ra became large. At Ra $= 8 \times 10^6$, $K_\psi = L_\omega \doteq 5 - 10$ with $M \doteq 15$. An attempt was made to develop near optimal sequences of the σ_ϕ's and the looping parameters but, again, the cost proved to be prohibitive. In addition, the parameters proved to be N-dependent.

Although the Galerkin method appears to offer an attractive alternative to finite difference methods, it should be noted that its apparent advantage with respect to computational efficiency is to some extent negated by a more tedious preparation of the problem and somewhat larger storage requirements. These disadvantages, of course, become more severe with increasing complexity of a given problem.

CONCLUSIONS

1. Conditionally stable ADI methods of solution for elliptic partial differential equations may successfully be applied to highly nonlinear, thermally driven flows. However, as the Rayleigh number becomes large,

(a) increasingly larger values of the ADI parameters (i.e., smaller values of the "effective" time step $\Delta t = \Delta^2/\sigma$) as well as increased compatibility between the flow and thermal fields at each "time step" are required to remove long period spurious oscillations in the dependent variables as the iterative scheme converges; and

(b) near optimal values of the relaxational parameters can in principal be obtained but their complex dependence on various idiosyncrasies of the nonlinear problem considered here does not admit generalization to other problems.

2. The Galerkin method (a particularly appropriate subset of the more general class of variational methods contained in the Method of Weighted Residuals; see e.g. [18, 19]) of solution is an attractive alternative to finite difference methods. For a given number of terms in the Galerkin expansion, the computational time required to advance solutions sequentially through a series of Rayleigh numbers is essentially independent of the magnitudes of the Rayleigh numbers. In contrast to finite difference methods, wall gradients are no less accurate than the internal field results themselves.

3. On a comparative basis, the Galerkin method is computationally more efficient than ADI methods, provided errors in such derived quantities as wall shear or heat flux in excess of 2–3 %, are acceptable. However, this advantage is offset by a laborious prepreparation of the algebraic problem and larger storage requirements and, all things considered, it is not clear which method is superior.

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